Int. J. Solids Structures Vol. 20, No. 3, pp. 245-255, 1984
Printed in Great Britain.
0020-7683/84 $3.00 + .00

A BOUNDARY-INTEGRAL EQUATION METHOD FOR BENDING OF ORTHOTROPIC PLATES

HANS IRSCHIK
Institute of Rational Mechanics, Technical University of Vienna, A-1040 Wien, Karlsplatz 13, Austria

(*Received* 1 *July* 1982; *in revised form* 24 *January* 1983)

Abstract—A boundary-integral equation method for bending of thin orthotropic linear elastic plates with polygonal planform under quasistatic pressure loading is presented. A change from Huber's differential equation for the deflection to the simpler biharmonic equation of Kirchoff's isotropic plate theory by means of a linear transformation of the plate domain renders a non-classical boundary value problem; the transformed plate is embedded in a basic rectangular domain; coincidence of boundaries is applied as far as possible. Using Green's matrix of the basic isotropic plate, a vectorial boundary integral equation for a fictitious density function vector is developed, which has to be defined on part of the actual boundary only. Solution generally is achieved by a numerical procedure, and Green's matrix is split into a regular and a singular part. Analytic integrals of the latter are independent of the special choice of the basic domain. Having solved the integral equation, all interesting state variables within the isotropic domain are evaluated by means of integration and may be transformed back to the original orthotropic plate in a simple manner. Method is tested in example problems.

1. INTRODUCTION

Following the well-known theory dating back to Huber ([1], see also [2], p. 364–369), this paper is concerned with thin, linear elastic, orthotropic plates under quasistatic pressure loading. Thereby, plates of convex polygonal planform and clamped, simply supported or free boundaries are considered. It is assumed, that the properties of the given elastic constants enable a change from Huber's differential equation for the deflection to the much simpler biharmonic equation of Kirchhoff's isotropic plate-theory ([2], p. 82) by means of a linear, orthogonal transformation of the given plate domain. This, e.g., works for two-way reinforced concrete plates and was used first by Huber himself[3] in case of simply supported rectangular plates.

For polygonal plates of arbitrary shape it is seen, that the character of kinematical boundary conditions is not changed by the transformation; for dynamical boundary conditions however, this is generally not true. Thus, a nonclassical boundary-value problem in the isotropic plate domain results. Only for edges parallel to the principal axes of orthotropy agreement with Kirchhoff's boundary conditions ([2], p. 84) remains.

Accordingly, the application of a boundary integral equation technique based on the method of Green's functions, which recently was developed for classical isotropic plates[4], seems to be a problem-oriented and convenient strategy. Thereby some ideas given by Melnikov[5] for problems of plane elasticity are paralleled once more.

The transformed polygonal plate is embedded in a rectangular domain with edges parallel to the principal axes of orthotropy, where coincidence of boundary conditions is applied as far as possible. This obviously is facilitated by usual types of plate constructions. Furthermore, rectangular plates with classical, homogeneous boundary conditions are treated in detail in literature ([2], p. 80–153).

Now, the unknown deflections and their derivatives in the transformed polygonal domain are represented by superposition of solutions in the basic rectangular plate, namely the solution due to the given loading and some homogeneous solutions. The latter correspond to a fictitious density function vector with components to be interpreted as line loads and moments distributed along the remaining, not coinciding part of the boundary. The boundary conditions there, being not yet satisfied, lead to a pair of coupled integral equations for the components of the density function vector, where the kernel is the corresponding Green's matrix of the rectangular domain.

Solution to the integral equations is achieved by a numerical procedure analogous to [4]. A system of linear equations is derived by stepwise integration over equidistant integration intervals. For that purpose, Green's functions are split into a regular and a singular part. The integrals over the singular part are independent of the choice of the special basic domain and, according to the nonclassical character of the boundary value problem, are listed below. Because boundary conditions partly are satisfied by means of the analytic formulation, the size of the system of equation generally is much smaller than that derived by standard numerical procedures.

Having solved the integral equations numerically, all the interesting state variables within the isotropic polygonal domain are explicitely known to about the same numerical accuracy and are retransformed to the actual orthotropic plate in a very simple manner.

Procedure is tested in example problems with simply supported basic domains.

2. A NONCLASSICAL BOUNDARY-VALUE PROBLEM FOR THE BIHARMONIC DIFFERENTIAL OPERATOR

In this paper thin, linear elastic plates of convex, polygonal planform under arbitrary pressure loading $\bar{p}$ are considered. Homogeneous orthotropic material is assumed, where one of the three perpendicular planes of elastic symmetry is the midplane of the plate. Then Huber's differential equation for the plate deflection $\bar{w}$ is valid ([1], see also [2], pp. 364–369):

$$K_x\bar{w}_{,xxxx} + 2H\bar{w}_{,xxyy} + K_y\bar{w}_{,yyyy} = \bar{p}. \tag{1}$$

Herein (x, y) denotes a global cartesian coordinate system of the midplane with axes parallel to the principal axes of orthotropy; the bar refers to functions of the given orthotropic plate domain. Differential relations between $\bar{w}$ and moments $\bar{m}$ or shearing forces $\bar{q}$ are given in Appendix A for convenience, where the four necessary elastic constants are denoted by K_x,K_y,K_{xy} and H. Appropriate definitions of these stiffness factors may be taken from the literature in numerous cases (e.g. [2], pp. 364–369) and are cited in Appendix 1 for two-way reinforced concrete plates.

It is well known (and first was noted by Huber himself in [3]), that eqn (1) may linearly be transformed to the governing biharmonic equation of Kirchhoff's isotropic plate-theory ([2], p. 82) under certain circumstances. This remarkable advantage is used for simply supported rectangular plates only in the literature (e.g. [6], p. 313). Here, polygonal plates with simply supported, clamped or free boundaries are treated (see Fig. 1) and the influence of the transformation upon the formulation of the boundary conditions will be studied in detail.

For that purpose the prescribed kinematical or dynamical conditions at the boundary C of the plate, forming a vector $\mathbf{Z}$ of two components and being defined in a local (n, s)-coordinate system on C (see Fig. 1), have to be represented in the global coordinate system by means of deflection $\bar{w}$ and its derivatives. This is performed in Appendix 2, Table 1.

Now, according to Huber's cited observation, the given orthotropic plate domain is subjected to the following linear transformation:

$$x = x', y = \epsilon y', \epsilon = (K_y/K_x)^{1/4}; \tag{2.1}$$

$$\bar{w}(x, y) = \bar{\bar{w}}(x', y'), \bar{p}(x, y) = \bar{\bar{p}}(x', y'). \tag{2.2}$$

Double bars denote functions of the transformed domain and the new global coordinates are (x', y').

Using

$$\bar{w}_{,x^jy^k} = \epsilon^{-k}\bar{\bar{w}}_{,x'^jy'^k}; \tag{3}$$

$$j, k = 0, \ldots, 4; \quad j + k \leq 4,$$

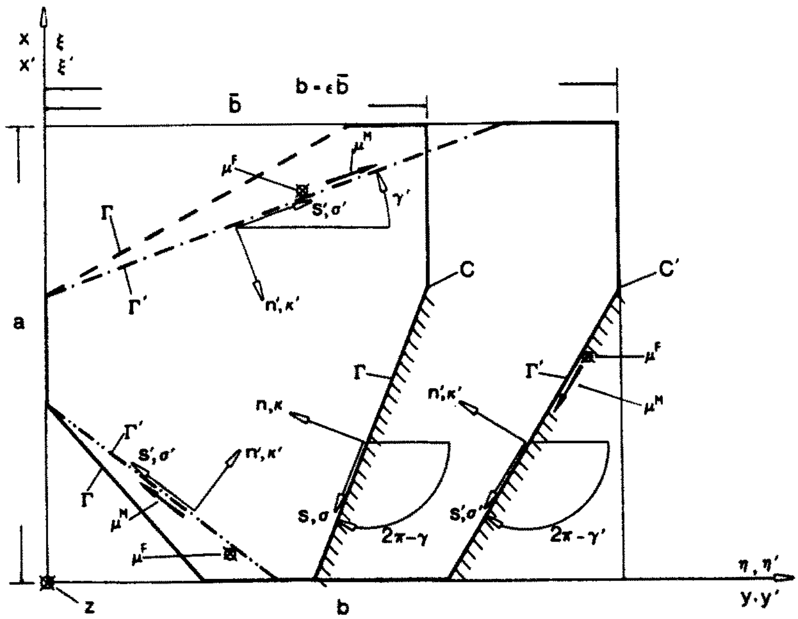

Fig. 1. Geometry of polygonal, transformed polygonal and basic domain. Definition of density functions along Γ' and coordinate systems used in the general formulation. clamped, —— simply supported, —— free, —·—·— transformed simply supported, ---- transformed free edge.

it is seen, that for $H^2 = K_x K_y$, eqn (1) becomes:

$$\Delta'\Delta'\bar{w} = \bar{p}/K; \quad \Delta'(\ ) = (\ )_{,x'x'} + (\ )_{,y'y'}. \tag{4}$$

This is the desired biharmonic equation for the deflection, considerably simplifying the solution of the problem. $K = K_x$ is the fictitious plate stiffness of the transformed, isotropic plate domain.

Clearly, eqn (4) could have been derived by modifying both of the global coordinates instead of one only. In every case, however, it has to be required, that $H^2 = K_x K_y$. This, e.g. is the case for the already mentioned two-way reinforced concrete plate (see Appendix 1), often is used exclusively in the literature (e.g. [7]), and will be assumed here, too.

By means of the transformation (2) the state vector $\mathbf{Z}$ is changed to $\bar{\mathbf{Z}}$. The latter is given in Appendix 2, Table 2, in global coordinates (x', y') and is represented in Table 3 in local coordinates (n', s') of the transformed boundary C' (see Fig. 1.).

As it is seen, the kinematical boundary conditions may be represented in accordance to Kirchhoff's theory ([2], pp. 83, 84). For dynamical boundary conditions this in general is not true. In case of skew edges with dynamical boundary conditions, Table 3 and eqn (4) define a nonclassical boundary value problem for the biharmonic differential operator in $\bar{w}$. For parts of C', which are parallel to the principal axes (x', y') accordance may be achieved by choosing Poisson's ratio ν' of the transformed, isotropic plate domain to be $\nu' = (H - 2K_{xy})/2$ in case of free edges.† ($H - 2K_{xy} \geq 0$ on physical reasons, see [2], p. 365). It should be noted, that the formulation of the transformed boundary conditions is by no means more difficult than in the original problem (compare Tables 1 and 2), the new governing differential eqn (4) being of a much more convenient form than Huber's equation (1).

Having solved the new problem stated above, all kinematical and dynamical functions of the original, orthotropic domain may be obtained by means of a simple re-transformation, which is also given in Appendix 1.

†Compare, however, ([6], p. 314), where it is erroneously stated, that for orthotropic rectangular plates with two free opposite edges, the other edges being simply supported, no use can be made of isotropic plate solutions.

3. A BOUNDARY-INTEGRAL EQUATION METHOD FOR THE TRANSFORMED PROBLEM USING GREEN'S FUNCTIONS OF FINITE DOMAINS

Solutions of the problem defined by eqn (4) and Table 3 generally have to be obtained through numerical methods.

In that connection the application of a new boundary-integral equation technique, which is based on the method of Green's functions and has been developed successfully for classical isotropic Kirchhoff-plates in [4],† seems to be a convenient strategy and will be worked out below.

At first stage, the transformed polygonal plate (see eqn 2) is embedded in a basic rectangular isotropic plate with classical homogeneous boundary conditions. The latter problem is well treated in the literature with respect to arbitrary loadings and Green's functions. The edges of the basic plate are situated parallel to the transformed axes of orthotropy (x', y'). Coincidence of boundaries is applied as far as possible, which is realized in such a manner, that boundary conditions become identical there (note Table 3). They will be satisfied exactly at those boundaries throughout the numerical solution. (Obviously in most cases of technical importance some edges of the polygonal orthotropic plate will be parallel to the principal axes of orthotropy for technical reasons.)

Now, the unknown deflection $\bar{w}$ will consist of two parts, both corresponding to the basic rectangular plate. The first part w is due to the given loading $\bar{p}$, which conveniently may be extended to the exterior of the polygonal domain to form p. Thus,

$$w(x', y') = \int_{\xi'=0}^{a} \int_{\eta'=0}^{b} w^F(x', y'; \xi', \eta') p(\xi', \eta') \, d\xi' \, d\eta', \tag{5}$$

where w^F denotes the Green's function for a unit force $F = 1$. The second part w_h consists of homogeneous solutions in order to satisfy the boundary conditions along the line Γ' of the actual boundary not already coinciding with the rectangular edges:

$$w_h(x', y') = \int_{\Gamma'} w^F(x', y'; \sigma') \mu^F(\sigma') \, d\sigma' + \int_{\Gamma'} w^M(x', y'; \sigma') \mu^M(\sigma') \, d\sigma'. \tag{6}$$

Herein σ' is the arclength of Γ', and μ^F, μ^M denote unknown line load densities of external force and moment distributions along Γ', respectively. For coordinate systems and definitions see Fig. 1.

The Green's function w^M is the deflection of the rectangular plate due to a unit couple $M = 1$ with moment vector oriented parallel to the boundary Γ' and may be evaluated from w^F according to [9]:

$$w^M(x', y'; \xi', \eta') = w^F(\xi', \eta'; x', y')_{,\varkappa'}, \tag{7}$$

where $\xi' = \xi'(\sigma', \varkappa')$ and $\eta' = \eta'(\sigma', \varkappa')$, see Fig. 1.

The fictitious line load density functions are to be determined from the coincidence of the deflection $w + w_h$ of the rectangular plate with the deflection $\bar{w}$ in the polygonal plate. Thus, the boundary conditions on Γ', which form a vector $\bar{\mathbf{Z}}$ of two components (see Tables 2 and 3), have to be satisfied at the inner site $n' = 0_+$, where n' is the inner normal coordinate of Γ'. This leads to a boundary vector integral equation for the line-load density functions, which may be gathered to the vectorial form $\mathbf{f}^T = [\mu^F, \mu^M]$:‡

$$\bar{\mathbf{Z}}(s') = \mathbf{Z}(s') + \int_{\Gamma'} \mathbf{G}(s'; \sigma') \mathbf{f}(\sigma') \, d\sigma' \Big|_{n'=0_+}, \tag{8}$$

†Reference [4] and this paper is influenced by some ideas given in Melnikov's work [5] on plane problems of elasticity.

‡$\mathbf{f}^T$ denotes the transpose of $\mathbf{f}$.

where (s') and (σ') denote arclength along Γ', and the 2×2 matrix $\mathbf{G}$ is Green's matrix corresponding to the boundary conditions $\bar{\mathbf{Z}}$. The vector $\mathbf{Z}$ denotes the corresponding state along Γ' in the rectangular plate due to the loading p.

The numerical solution of eqn (8), which may contain singular and non-symmetric kernels, analogously follows the method given in [4] for classical isotropic plates. Hence, the boundary Γ' is divided into l equidistant intervals of length λ and the unknown line load density vector becomes stepwise constant in the intervals. By collocation the boundary conditions are fulfilled pointwise in the midth of the intervals, s_i' $(i = 1, \ldots, l)$. Hence, a set of $2l$ linear equations for the $2l$ components of the density vectors $\mathbf{f}_j (j = 1, \ldots, l)$ is obtained:

$$\bar{\mathbf{Z}}(s_i') = \mathbf{Z}(s_i') + \lambda \sum_{\substack{j=1 \\ j \neq i-\beta}}^{l} \mathbf{G}(s_i'; s_j')\mathbf{f}_j + \sum_{j=i-\alpha}^{j=i+\alpha} \int_{s_j'-(\lambda/2)}^{s_j'+(\lambda/2)} \mathbf{G}(s_i', n'; \sigma')\,\mathrm{d}\sigma' \bigg|_{n'=0_+} \mathbf{f}_j, \quad -\alpha \leq \beta \leq \alpha, \tag{9}$$

where in most of the nonsingular intervals $(j \neq i - \beta)$ the integrals of eqn (8) are replaced by finite sums using the rectangular formula and $\mathbf{G}$ is evaluated by means of the well-known single series representations of Green's functions for rectangular plates (see e.g. [2, 6 or 10]; for a proper formulation of a fast convergence of the series see [4]). In the (possibly) singular intervals and in close neighborhood $j = i - \beta$, $-\alpha \leq \beta \leq \alpha$, Green's matrix $\mathbf{G}$ is split into a regular part $\mathbf{G}_R$ and a singular part $\mathbf{G}_S$. The latter is derived by proper order differentiation from the fundamental solutions of the infinite plate domain (see e.g. [2], p. 325):

$$w_\infty{}^F = \frac{r^2}{8\pi K} \ln \frac{r}{b}, \quad r = [(\sigma' - s')^2 + (x' - n')^2]^{1/2}, \tag{10}$$

and (see eqn 7):

$$w_\infty{}^M = \frac{r}{4\pi K} \ln \frac{r}{b} \cos \varphi, \quad \varphi = \operatorname{arctg} \frac{\sigma' - s'}{x' - n'}. \tag{11}$$

$\mathbf{G}_S$ is listed in Appendix 3, Table 4. In the intervals $j = i - \beta$, $-\alpha \leq \beta \leq \alpha, j \neq i$, the regular part $\mathbf{G}_R$ is computed from $\mathbf{G}_S$ and the single series solutions of $\mathbf{G}$: $\mathbf{G}_R = \mathbf{G} - \mathbf{G}_S$. In the singular intervals $i = j$ $\mathbf{G}_R$ then is evaluated by numerical interpolation.

Now the integrals of eqn (9) may be worked out:

$$\int_{s_j'-(\lambda/2)}^{s_j'+(\lambda/2)} \mathbf{G}(s_i', n'; \sigma')\,\mathrm{d}\sigma' \bigg|_{n=0_+} = \int_{s_j'-(\lambda/2)}^{s_j'+(\lambda/2)} \mathbf{G}_S(s_i', n'; \sigma')\,\mathrm{d}\sigma' \bigg|_{n=0_+} + \lambda \mathbf{G}_R(s_i'; s_j'). \tag{12}$$

Results of analytical integration of $\mathbf{G}_S$ are listed in Appendix 3, Table 5. Being independent from the special choice of the basic domain, Table 5 ensures the generality of the problem-oriented method.

The linear system of equations, (9) is generally well behaved and may be solved by standard procedures. For further numerical treatments, e.g. choice of l and α or establishment of fast convergence of the series solutions, see [4]. From there, results may be transferred directly with respect to the equal order of derivatives of w^F and w^M of Green's matrix $\mathbf{G}$.

$\bar{w}$ and its derivatives in the interior of the transformed polygonal plate are calculated by numerical integration:

$$\bar{w}_{,x'^j y'^k}(x', y') = w_{,x'^j y'^k}(x', y') + \sum_{i=1}^{l} \int_{s_i'-(\lambda/2)}^{s_i'+(\lambda/2)} \mathbf{e}^T(x', y'; \sigma')\,\mathrm{d}\sigma' \mathbf{f}_i, \tag{13}$$

$$e^T = (w^F_{,x'^j y'^k}, w^M_{,x'^j y'^k}).$$

In eqn (13), numerical integration by rectangular-formula is appropriate. Only for points (x', y') in close distance to the boundary line Γ', especially for higher-order singularities in e, integration has to be performed analogously to eqns (9) and (12).

In connection with the re-transformation given in Appendix 1, eqn (13) then determines all kinematical and dynamical components of the state vector in the original orthotropic plate.

4. EXAMPLE PROBLEMS

To test the method, a FORTRAN program was set up and implemented at the CYBER 74 computer of the Technical University of Vienna. Various calculations were carried out, showing both the low numerical effort and the high accuracy of the numerical procedure. As an example, dimensionless results for rectangular plates with two edges simply supported $(y = 0, b)$ and two edges free $(x = 0, \bar{a})$ under constant pressure loading are presented in Table 6 and compared with solutions for isotropic plates published in ([2], p. 219). At first, a square plate with $K_x/K_y = 16$ and $K_x/K_y = 1/16$ is studied; according to the remarks given above, the results must correspond to those of an isotropic plate with $\bar{b}/\bar{a} = 2$ or $\bar{b}/\bar{a} = 1/2$, respectively. Secondly, results for orthotropic plates with $\bar{b}/\bar{a} = \sqrt{2}$, $K_x/K_y = 1/4$ and $\bar{b}/\bar{a} = 1/\sqrt{2}$, $K_x/K_y = 4$ are given, both corresponding to a square isotropic plate. Boundary conditions are exactly satisfied at simply supported edges and collocatively at the free edges. The free-edge boundary condition has been chosen to show the functioning of the method in case of highest possible order singularities of Green's matrix. A simply supported rectangular plate is used as basic domain and each free edge is divided into $l = 20$ intervals in Table 6. Test calculations show, however, that the method renders satisfactory numerical results for a much more larger length λ of the intervals, too.

Furthermore, dimensionless results for a $(\gamma = \pi/3)$ skew orthotropic plate of rhombic planform with two edges simply supported and two edges free under constant pressure loading are given in Fig. 2. Having in mind reinforced concrete plates, Poisson's ratio of the transformed plate is chosen to be $\nu = 0.2$: thus $K_{xy} = 0.4\,H$. In a range of technical importance K_x/K_y is considered to vary between 1/4 and 4/1. The simply supported edges parallel a principle axis of orthotropy,† and the boundary conditions there are satisfied exactly again.

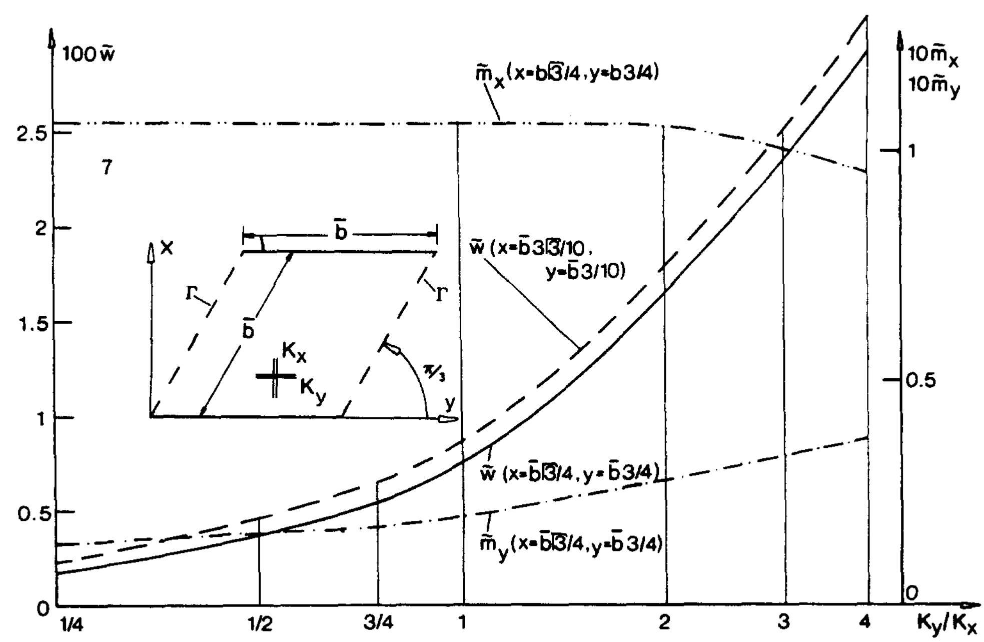

Fig. 2. Dimensionless deflections $\tilde{w} = K_y\bar{w}/\bar{p}_0\bar{b}^4$ and dimensionless bending moments $\tilde{m}_y = \bar{m}_y/\bar{p}_0\bar{b}^2$, $\tilde{m}_x = \bar{m}_x/\bar{p}_0\bar{b}^2$ for rhombic plates with two edges simply supported, two edges free and $\gamma = \pi/3$, $H = \sqrt{K_xK_y}$, $K_{xy} = 0.4H$ under constant pressure loading $\bar{p}_0$ for various values of K_y/K_x.

†This is a supplementary case to literature, see e.g. [11], where a principal axis parallels the free edges.

Results were compared to those derived by means of a standard Finite-Element program (SAPIV), computational costs of the boundary integral equation technique generally being substantial lower. That is especially the case for regions not neighbouring the center of the plate.

5. CONCLUDING REMARKS

Dealing with special orthotropy, Huber's differential equation of deflection is transformed to the biharmonic equation of Kirchhoff's isotropic theory by means of eqn (2). Transformed boundary conditions on straight lines are studied in Appendix C, and it is seen, that their formulation for dynamical conditions is equal to Kirchhoff's in case of edges parallel to the principal axes of orthotropy, and for kinematical conditions is equal in any case. This is used in the numerical solution of the new boundary value problem: extending a method for isotropic plates[4], a boundary integral equation technique using Green's matrices of basic finite isotropic domains is applied.† Now, boundary conditions may be satisfied exactly on parts of the boundary and the vectorial boundary integral equation (9) has to be established on the remaining parts only, thus decreasing further the number of unknowns and increasing accuracy. Integrals over singular parts of Green's matrix, being independent of the basic domain, are listed in Appendix 3. A simple computer program may then be designed, where the well-behaved system of eqns (9) is solved by standard procedures.

Deflections, moments and shearing forces of the original orthotropic plate are evaluated pointwise by means of numerical integration corresponding to eqn (13) and the simple re-transformation of Appendix 1. Computing costs are comparatively low. As usually observed in numerical methods, however, moments and shearing forces can not be evaluated accurately in the close vicinity of the not coinciding boundary Γ.

This work is part of the authors doctoral thesis[14], where computer realisations for the solution of the integral equations may be found also.

Acknowledgement—The author wishes to thank Prof. Dr. Franz Ziegler of the Technical University, Vienna, for many valuable discussions and comments with regard to the contents of this paper.

REFERENCES

1. M. T. Huber, Die Grundlagen einer rationellen Berechnung der kreuzweise bewehrten Eisenbetonplatten. Zeitschrift des Österr. *Ingenieur- und Architektenvereins* **30**, 557–564 (1914).
2. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*. 2nd Edn. McGraw-Hill, New York (1959).
3. M. T. Huber, Über die Biegung einer Rechteckplatte von ungleicher Biegesteifigkeit in der Längs- und Querrichtung. *Bauingenieur* **5**, 259–305 (1924).
4. H. Irschik and F. Ziegler, Application of the Green's Function Method to Thin Elastic Polygonal Plates. *Acta Mechanica* **39**, 155–169 (1981).
5. Yu. A. Melnikov, Some Applications of the Green's function method in mechanics. *Int. J. Solids Structures* **13**, 1045–1058 (1977).
6. K. G. Girkmann, *Flächentragwerke*. 6th Edn. Springer-Verlag, Wien (1974).
7. H. Olsen and F. Reinitzhuber, *Die Zweiseitig Gelagerte Platte*, Teil 1, 2. W. Ernst & Sohn, Berlin (1950).
8. F. Leonhardt, *Vorlesungen über Massivbau*, 3. Teil. Springer-Verlag, Berlin (1974).
9. P. Nemenyi, Eine neue Singularitätenmethode für die Elastizitätstheorie. *ZAMM* **9**, 488–490 (1929).
10. E. Bittner, *Platten und Behälter*. Springer-Verlag, Wien (1965).
11. J. B. Kennedy and D. S. R. Gupta, Bending of skew orthotropic plate structures. *J. Struct. Div., ASCE* **102**, 1559–1574 (1976).
12. M. Stern, A general boundary integral formulation for the numerical solution of plate bending problems. *Int. J. Solids Structures* **15**, 769–782 (1979).
13. B. C. Wu and N. J. Altiero, A boundary integral method applied to plates of arbitrary plan form. *Comput. Structures* **9**, 163–168 (1978).
14. H. Irschik, Anwendung der Methode der Green'schen Funktionen auf die Berechnung dünner, linear elastischer Platten mittels Randintegralgleichungen. Thesis, TU-Wien (1981).

†Hitherto existing boundary-integral equation methods usually employ Green's functions of the infinite domain, see e.g. [12] for isotropic plates. A boundary-integral equation method using Green's functions of a (finite) clamped circular plate was given in [13] for clamped isotropic plates.

APPENDIX 1

Basic relations of Huber's theory

In Huber's theory of orthotropic plates (eqn 1 and Table 1 of Appendix 2) the following differential relations between deflection $\bar{w}$ and moments $\bar{m}$ or shearing forces $\bar{q}$ are valid (see [2], p. 365):

$$\begin{aligned}\bar{m}_x &= -(K_x\bar{w}_{,xx} + (H - 2K_{xy})\bar{w}_{,yy}),\\ \bar{m}_y &= -(K_y\bar{w}_{,yy} + (H - 2K_{xy})\bar{w}_{,xx}),\\ \bar{m}_{xy} &= -2K_{xy}\bar{w}_{,xy};\\ \bar{q}_x &= -(K_x\bar{w}_{,xx} + H\bar{w}_{,yy})_{,x},\\ \bar{q}_y &= -(K_y\bar{w}_{,yy} + H\bar{w}_{,xx})_{,y}.\end{aligned}$$

(x, y) is a cartesian coordinate system of the midplan with axes parallel to the principal axes of orthotropy; the corresponding stiffness factors K_x, K_y, K_{xy} and H are given exemplarily for concrete plates with reinforcement in (x, y) directions ([2], p. 366):

$$\begin{aligned}K_x &= E_c(I_{xc} + (n-1)I_{xs})/(1 - \nu_c^2),\\ K_y &= E_c(I_{yc} + (n-1)I_{ys})/(1 - \nu_c^2),\\ H &= (K_xK_y)^{1/2},\\ K_{xy} &= (1 - \nu_c)H/2,\end{aligned}$$

where a linear law of material for steel as well as for concrete is assumed to be justified. c stands for concrete and s for steel; E denotes Young's modulus and ν Poisson's ratio; I is a moment of inertia with respect to the neutral plane; $n = E_s/E_c$.†

By means of the transformation (2) a new problem (eqn 4 and Table 2 or Table 3 of Appendix 2) is obtained. The corresponding relations between m-(x, y) or $\bar{q}(x, y)$ and the derivatives of $\bar{\bar{w}}(x', y')$ are:

$$\begin{aligned}\bar{m}_x &= -K(\bar{\bar{w}}_{,x'x'} + \nu\bar{\bar{w}}_{,y'y'}),\\ \bar{m}_y &= -H(\bar{\bar{w}}_{,y'y'} + \nu\bar{\bar{w}}_{,x'x'}),\\ \bar{m}_{xy} &= -(KH)^{1/2}(1 - \nu)\bar{\bar{w}}_{,x'y'};\\ \bar{q}_x &= -K\Delta'\bar{\bar{w}}_{,x'},\\ \bar{q}_y &= -(KH)^{1/2}\Delta'\bar{\bar{w}}_{,y'};\end{aligned}$$

$K = K_x$ and $\nu = (H - 2K_{xy})/H$ denote effective stiffness and Poisson's ratio of the transformed isotropic domain, respectively.

APPENDIX 2

Boundary conditions

Table 1. Table of homogeneous boundary conditions on (some part of) Γ, represented in the global (x, y)-coordinate system*

Boundary conditions on (some part of) C:	$\bar{\mathbf{Z}} = 0$
clamped:	$\begin{bmatrix}\bar{w}\\ \bar{w}_{,n}\end{bmatrix} = \begin{bmatrix}\bar{w}\\ -\bar{w}_{,x}\cos\gamma + \bar{w}_{,y}\sin\gamma\end{bmatrix} = 0$
simply supported:	$\begin{bmatrix}\bar{w}\\ \bar{m}_n\end{bmatrix} = \begin{bmatrix}\bar{w}\\ A\bar{w}_{,xx} + B\bar{w}_{,yy} + C\bar{w}_{,xy}\end{bmatrix} = 0$
free:	$\begin{bmatrix}\bar{m}_n\\ \bar{q}_n + \bar{m}_{ns,s}\end{bmatrix} = \begin{bmatrix}A\bar{w}_{,xx} + B\bar{w}_{,yy} + C\bar{w}_{,xy}\\ D\bar{w}_{,xxx} + E\bar{w}_{,xxy} + F\bar{w}_{,xyy} + G\bar{w}_{,yyy}\end{bmatrix} = 0$

where:

$$\begin{aligned}A &= -(K_x\cos^2\gamma + (H - 2K_{xy})\sin^2\gamma),\\ B &= -(K_y\sin^2\gamma + (H - 2K_{xy})\cos^2\gamma),\\ C &= 2K_{xy}\sin\gamma;\\ D &= \frac{K_x}{2}(3 - \cos 2\gamma)\cos\gamma + K_{xy}\sin\gamma\sin 2\gamma - \frac{H}{2}\sin\gamma\sin 2\gamma\\ E &= \frac{K_x}{2}\cos\gamma\sin 2\gamma + K_{xy}(1 + 3\cos 2\gamma)\sin\gamma - \frac{H}{2}(3 + \cos 2\gamma)\sin\gamma,\\ F &= -\frac{K_y}{2}\sin\gamma\sin 2\gamma - K_{xy}(1 - 3\cos 2\gamma)\cos\gamma + \frac{H}{2}(3 - \cos 2\gamma)\cos\gamma,\\ G &= -\frac{K_y}{2}(3 + \cos 2\gamma)\sin\gamma - K_{xy}\cos\gamma\sin 2\gamma + \frac{H}{2}\cos\gamma\sin 2\gamma.\end{aligned}$$

*Calculated by means of geometrical transformations according to Fig. 1, formulas of Appendix 1 and the following relations: $\bar{m}_n = \bar{m}_x\cos^2\gamma + m_y\sin^2\gamma - m_{xy}\sin 2\gamma$; $\bar{m}_{ns} = (1/2)(\bar{m}_y - \bar{m}_x)\sin 2\gamma - \bar{m}_{xy}\cos 2\gamma$; $\bar{q}_n = -\bar{q}_x\cos\gamma + \bar{q}_y\sin\gamma$; compare ([2], pp. 86, 87).

†Often the results of Kirchhoff's isotropic plate theory are used for the design of two-way reinforced concrete plates according to nonlinear laws of material, see e.g. ([8], p. 82); consideration of orthotropic behaviour may improve this strategy.

Table 2. Table of transformed boundary conditions (on some part of) Γ', represented in the global (x', y')-system†

Boundary conditions on (some part of) C:	$\bar{\bar{Z}} = 0$ on C':
clamped:	$\begin{bmatrix} \bar{\bar{w}} \\ -\bar{\bar{w}}_{,x'}\cos\gamma + w_{,y'}\epsilon^{-1}\sin\gamma \end{bmatrix} = 0$
simply supported:	$\begin{bmatrix} \bar{\bar{w}} \\ A\bar{\bar{w}}_{,x'x'} + B\epsilon^{-2}\bar{\bar{w}}_{,y'y'} + C\epsilon^{-1}\bar{\bar{w}}_{,x'y'} \end{bmatrix} = 0$
free:	$\begin{bmatrix} A\bar{\bar{w}}_{,x'x'} + B\epsilon^{-2}\bar{\bar{w}}_{,y'y'} + C\epsilon^{-1}\bar{\bar{w}}_{,x'y'} \\ D\bar{\bar{w}}_{,x'x'x'} + E\epsilon^{-1}\bar{\bar{w}}_{,x'x'y'} + \\ F\epsilon^{-2}\bar{\bar{w}}_{,x'y'y'} + G\epsilon^{-3}\bar{\bar{w}}_{,y'y'y'} \end{bmatrix} = 0$

where $\epsilon = (K_y/K_x)^{1/4}$.

†Derived from eqn (3) and Table 1; for coordinate systems and definitions see Fig. 1.

Table 3. Table of transformed boundary conditions (on some part of) Γ', represented in the local (n', s')-system†

boundary conditions on (some part of) C:	$\bar{\bar{Z}} = 0$ on C':
clamped:	$\begin{bmatrix} \bar{\bar{w}} \\ \bar{\bar{w}}_{,n'} \end{bmatrix} = 0$
simply supported:	$\begin{bmatrix} \bar{\bar{w}} \\ A'\bar{\bar{w}}_{,n'n'} + C'\bar{\bar{w}}_{,n's'} \end{bmatrix} = 0$
free:	$\begin{bmatrix} A'\bar{\bar{w}}_{,n'n'} + B'\bar{\bar{w}}_{,s's'} + C'\bar{\bar{w}}_{,n's'} \\ D'\bar{\bar{w}}_{,n'n'n'} + E'\bar{\bar{w}}_{,n'n's'} + \\ F'\bar{\bar{w}}_{,n's's'} + G'\bar{\bar{w}}_{,s's's'} \end{bmatrix} = 0$

where:

$$A' = A\cos^2\gamma' + B\epsilon^{-2}\sin^2\gamma' - \frac{C}{2}\epsilon^{-1}\sin 2\gamma',$$

$$B' = A\sin^2\gamma' + B\epsilon^{-2}\cos^2\gamma' + \frac{C}{2}\epsilon^{-1}\sin 2\gamma',$$

$$C' = -A\sin 2\gamma' + B\epsilon^{-2}\sin 2\gamma' - C\epsilon^{-1}\cos 2\gamma',$$

$$D' = -D\cos^3\gamma' + \frac{E}{2}\epsilon^{-1}\cos\gamma'\sin 2\gamma' - \frac{F}{2}\epsilon^{-2}\sin\gamma'\sin 2\gamma' + G\epsilon^{-3}\sin^3\gamma',$$

$$E' = \frac{3}{2}D\cos\gamma'\sin 2\gamma' + E\epsilon^{-1}(\cos\gamma'\cos 2\gamma' - \frac{1}{2}\sin\gamma'\sin 2\gamma') -$$
$$- F\epsilon^{-2}(\sin\gamma'\cos 2\gamma' + \frac{1}{2}\cos\gamma'\sin 2\gamma') + \frac{3}{2}G\epsilon^{-3}\sin\gamma'\sin 2\gamma',$$

$$F' = -\frac{3}{2}D\sin\gamma'\sin 2\gamma' - E\epsilon^{-1}(\sin\gamma'\cos 2\gamma' + \frac{1}{2}\cos\gamma'\sin 2\gamma') -$$
$$- F\epsilon^{-2}(\cos\gamma'\cos 2\gamma' - \frac{1}{2}\sin\gamma'\sin 2\gamma') + \frac{3}{2}G\epsilon^{-3}\cos\gamma'\sin 2\gamma',$$

$$G' = D\sin^3\gamma' + \frac{E}{2}\epsilon^{-1}\sin\gamma'\sin 2\gamma' + \frac{F}{2}\epsilon^{-2}\cos\gamma'\sin 2\gamma' + G\epsilon^{-3}\cos^3\gamma',$$

and $\epsilon = (K_y/K_x)^{1/4}$.

†Derived from equation (3) and Jacobian matrix with respect to Fig. 1:

$$\begin{bmatrix} (\ )_{,x'} \\ (\ )_{,y'} \end{bmatrix} = \begin{bmatrix} \sin\gamma' & -\cos\gamma' \\ \cos\gamma' & \sin\gamma' \end{bmatrix} \cdot \begin{bmatrix} (\ )_{,n'} \\ (\ )_{,s'} \end{bmatrix}.$$

APPENDIX 3

Singular parts of Green's matrices and their integrals

Table 4. Table of singular components of G†

Boundary conditions on (some part of) Γ	$8\pi K G_s$	
simply supported:	0	0
	$A'\left(2\ln\frac{r}{b'}+\cos 2\varphi+2\right)+C'\sin 2\varphi$	$2A'\frac{\cos\varphi}{r}(2-\cos 2\varphi)-2C'\frac{\sin\varphi}{r}\cos 2\varphi$
free:	$A'\left(2\ln\frac{r}{b'}+\cos 2\varphi+2\right)+C'\sin 2\varphi$ $+B'\left(2\ln\frac{r}{b'}+2-\cos 2\varphi\right)$ $-D'\frac{\cos\varphi}{r}(2-\cos 2\varphi)+E'\frac{\sin\varphi}{r}\cos 2\varphi$ $-F'\frac{\cos\varphi}{r}\cos 2\varphi-G'\sin\varphi\,(2+\cos 2\varphi)$	$2A'\frac{\cos\varphi}{r}(2-\cos 2\varphi)-2C'\frac{\sin\varphi}{r}\cos 2\varphi$ $+2B'\frac{\cos\varphi}{r}\cos 2\varphi$ $-D'\frac{1}{r^2}(\cos 4\varphi-2\cos 2\varphi)+E'\frac{\sin 2\varphi}{2}$ $\times\left(1-2\cos 2\varphi\right)+F'\frac{1}{r^2}\cos 4\varphi$ $+G'\frac{\sin 2\varphi}{r^2}\left(1+2\cos 2\varphi\right)$

†Calculated from Table 3 and eqns (10) and (11).

$r=[(\sigma'-s')^2+(\varkappa'-n')^2]^{1/2},\ \varphi=\operatorname{artg}\frac{\sigma'-s'}{\varkappa'-n'}.$

Table 5. Integrals over singular components of G in the interval $j=i-\beta$†

Boundary conditions on (some part of) Γ	$4\pi\lambda\int_{s_j'-(\lambda/2)}^{s_j'+(\lambda/2)} G_s(s_i',n';\sigma')\,d\sigma'\Big\vert_{n'=0_+}$	
simply supported:	0	0
	$A'\lambda^2\left[\ln\left(\frac{\lambda}{2b'}\sqrt{q_1q_2}\right)-\frac{1}{2}+\lvert\beta\rvert\ln q_1/q_2\right]$	$-2A'\pi\lambda q_3+C'\lambda\ln q_1/q_2$
free:	$A'\lambda^2\left[\ln\left(\frac{\lambda}{2b'}\sqrt{q_1q_2}\right)-\frac{1}{2}+\lvert\beta\rvert\ln q_1/q_2\right]$ $+B'\lambda^2\left[\ln\left(\frac{\lambda}{2b'}\sqrt{q_1q_2}\right)+\frac{1}{2}+\lvert\beta\rvert\ln q_1/q_2\right]$	$-2A'\pi\lambda q_3+C'\lambda q_5\ln q_1/q_2$
	$D'\pi\lambda q_3-E'\frac{\lambda}{2}q_5\ln q_1/q_2-G'\frac{\lambda}{2}q_5\ln q_1/q_2$	$(6D'/q_1q_2-2F'/q_1q_2)q_4$
where:	$q_1=1+2\lvert\beta\rvert,\quad q_2=\lvert 1-2\lvert\beta\rvert\rvert,$ $q_3=\begin{cases}0\ldots\ldots\beta\neq 0\\1\ldots\ldots\beta=0\end{cases},\quad q_4=\begin{cases}-1\ldots\ldots\beta\neq 0\\ 1\ldots\ldots\beta=0\end{cases},\quad q_5=\begin{cases}-1\ldots\ldots\beta<0\\+1\ldots\ldots\beta>0\end{cases},0\ldots\beta=0.$	

†Calculated from Table 4, where the limit $n=0_+$ was applied after integration.

Table 6. Dimensionless deflections $K_x\bar{w}/\bar{p}_0\bar{b}^4$ and moments $\bar{m}_y\sqrt{K_x/K_y}/\bar{p}_0\bar{b}^2$ of rectangular plates under constant pressure loading $\bar{p}_0$ for various values of K_x/K_y and $\bar{a}/\bar{b}$. $H=\sqrt{K_xK_y}$, $K_{xy}=0.35\ H$

		$x=\bar{a}/2$	$y=\bar{b}/2$	$x=\bar{a}$	$y=\bar{b}/2$
		$\frac{K_x\bar{w}}{p_0\bar{b}^4}$	$\frac{\bar{m}_y}{p\bar{b}^2}\sqrt{\frac{K_x}{K_y}}$	$\frac{K_x\bar{w}}{p_0\bar{b}^4}$	$\frac{\bar{m}_y}{p_0\bar{b}^2}\sqrt{\frac{K_x}{K_y}}$
$\bar{b}/\bar{a}=1$,	$I=20$:	0.01296	0.1244	0.01533	0.1311
	[2]:	0.01289	0.1235	0.01521	0.1329
$K_x/K_y=16$	$\delta\%$:	0.54	0.72	0.81	−1.35
$\bar{b}/\bar{a}=1$,	$I=20$:	0.01385	0.1243	0.01455	0.1243
	[2]:	0.01377	0.1235	0.01443	0.1259
$K_x/K_y=1/16$	$\delta\%$:	0.58	0.65	0.83	−1.27
$\bar{b}/\bar{a}=\sqrt{2}$, $K_x/K_y=1/4$	$I=20$:	0.01313	0.1231	0.01498	0.1335
and	[2]:	0.01309	0.1225	0.01509	0.1318
$\bar{b}/\bar{a}=1/\sqrt{2}$, $K_x/K_y=4$	$\delta\%$:	0.31	0.49	−0.79	1.29